How We Think
A Presentation for IACME XIII and SIEMAT III

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Overview

My main goal for this paper is to introduce a theory of decision-making – a theory that provides a way of explaining how and why people make the decisions they do, in the middle of complex activities such as teaching. My current research has evolved from my earlier problem solving work on problem solving (Schoenfeld, 1985), so to set the stage for this discussion I will briefly describe that work – what it showed, and the questions it did not answer. That will allow me to describe what a complete theory should be able to accomplish. I then turn to the main body of this paper, three studies of teaching. In those examples I show how, under certain circumstances, it is possible to *model* the act of teaching, to the point where one can provide a grounded explanation of every decision that a teacher makes during an extended episode of teaching. Following that, I give some other examples to show that the theory is general, and I make a few concluding comments.

Introduction

I am honored to have been invited to give presentations at the *XIII Interamerican Conference on Mathematics Education* (IACME XIII) and the *International Seminar of Mathematics Education* (SIEMAT III), and grateful to the organizers for inviting me. It is a great pleasure to visit my friends and colleagues in Brazil and the Americas.

These invitations provided me with a wonderful opportunity to set a personal goal. When Angel Ruiz and I first talked two years ago about my coming to Brazil, I was working on my book *How We Think.* I promised him – and myself – that I would be able to bring a published copy of the book to this conference, and that I would talk about the book.

I keep my promises! I am happy to say that the book *How We Think* now exists (Schoenfeld, 2010). Here is what I am going to discuss.

The Challenge

Suppose that you are in the middle of some "well-practiced" activity, something you have done often so that it is familiar to you. Depending on who you are, it might be

- cooking a meal

- fixing a car

- teaching a class

- doing medical diagnosis or brain surgery.

The challenge is this: If I know "enough" about you, can I explain (i.e., build a cognitive model that explains) every single action you take and every decision you make?

My goal for this paper, and for my talk, is to describe an analytic structure that does just that – an analytic structure that explains how and why people act the way they do, on a moment-by-moment basis, in the midst of complex, often social activities.

My major claim is this:

People's in-the-moment decision making is a function of their knowledge and resources, goals, and beliefs and orientations. Their decisions and actions can be "captured" (explained and modeled) in detail using only these constructs.

The main substance of this paper (as in the book) consists of three analyses of teaching, to convey the flavor of the work. Of course, it is no accident that I chose mathematics teaching as the focal area for my analyses. I am, after all, a mathematics educator! But more to the point, teaching is a knowledge-intensive, highly interactive, dynamic activity. If it is possible to validate a theory that explains teachers' decision-making in a wide range of circumstances, then that theory should serve to explain all well practiced behavior.

Background: Problem Solving

As I noted above, my current work is an outgrowth of my earlier research on mathematical problem solving. Here I want to summarize the "bottom lines" of that work, to show how it lays the groundwork for my current research.

My major argument about mathematical problem solving (see Schoenfeld, 1985, for detail) was that it is possible to explain someone's success or failure in trying to solve problems on the basis of these 4 things:

1. *Knowledge (or more broadly, resources).* This is not exactly shocking – but, knowing what knowledge and resources a problem solver has *potentially* at his or her disposal is important.

2. *Problem solving strategies, also known as "heuristics."* We know from Pólya's work that mathematicians use such strategies, "rules of thumb for making progress when you do not know a direct way to a solution." Faculty use them; they pick them up by themselves, by experience. Typically, students don't use them. But, my research showed that students can learn to use them.

3. *"Metacognition," or "Monitoring and self-regulation."* Effective problem solvers plan and keep track of how well things are going as the implement their plans. If they seem to be making progress, they continue; if there are difficulties, they re-evaluate and consider alternatives. Ineffective problem solvers (including most students) do not do this. As a result, they can fail to solve problems that they *could* solve. Students can learn to be more effective at these kinds of behaviors.

4. *Beliefs.* Students' beliefs about themselves and the nature of the mathematical enterprise, derived from their experiences with mathematics, shape the very knowledge they draw upon during problem solving and the ways they do or do not use that knowledge. For example, students who believe that "all problems can be solved in 5 minutes or less" will stop working on problems even though, had they persevered, they might have solved them. Students who believe that "proof has nothing to do with discovery or invention" will, in the context of "discovery" problems, make conjectures that contradict results they have just proven. (See Schoenfeld, 1985).

In sum: By 1985 we know what "counted" in mathematical problem solving, in the sense that we could explain, *post hoc*, what accounted for success or failure. As the ensuing 25 years have shown, this applied to all "goal-oriented" or problem solving domains, including mathematics, physics, electronic trouble-shooting, and writing.

BUT… There was a lot that this framework I have just described didn't do. In the research I conducted for *Mathematical Problem Solving*, people worked in isolation on problems that I gave them to solve. Thus,

- The goals were established (i.e., "solve this problem").

- The tasks didn't change while people worked on them.

- Social interactions and considerations were negligible.

In addition, MPS offered a *framework*, not a theory. Above and beyond pointing out what is important – that is what a framework does – a theory should provide rigorous explanations of how and why things fit together. That is what my current work is about. What I have been working on for the past 25 years is a theoretical approach that explains:

- how and why people make the choices they do,

- while working on issues they care about and have some experience with,

- amidst dynamically changing social environments.

I can think of no better domain to study than teaching. Teaching is knowledge-intensive. It calls for instant decision-making in a dynamically changing environment. It's highly social. And if you can model teaching, you can model just about anything! I will argue that if you can model teaching, you can model: shopping; preparing a meal; an ordinary day at work; automobile mechanics; brain surgery (or any other medical practice), and other comparably complex, "well practiced" behaviors. All of these activities involve goal-oriented behavior – drawing on available resources (not the least of which is knowledge) and making decisions in order to achieve outcomes you value.

The goal of my work, and this paper, is to describe a theoretical architecture that explains people's decision-making during suh activities.

How Things Work

My main theoretical claim is that goal-oriented "acting in the moment" – including problem solving, tutoring, teaching, cooking, and brain surgery – can be explained and modeled by a theoretical architecture in which the following are represented: Resources (especially knowledge); Goals; Orientations (an abstraction of beliefs, including values, preferences, etc.); and Decision-Making (which can be modeled as a form of subjective cost-benefit analysis). For substantiation, in excruciating detail, please see my book, *How we Think.* To briefly provide substantiation I will provide three examples in what follows. But first, a top-level view of how things work is given in Figure 1. The basic structure is recursive: Individuals orient to situations and decide (on the basis of beliefs and available resources) how to pursue their goals. If the situation is familiar, they implement familiar routines; if things are unfamiliar or problematic, they reconsider. It may seem surprising, but if you know enough about an individual's resources, goals, and beliefs, this approach allows you to model their behavior (after a huge amount of work!) on a line-by-line basis.

How Things Work

- An individual enters into a particular context with a specific body of resources, goals, and orientations.

- The individual takes in and orients to the situation. Certain pieces of information and knowledge become salient and are activated.

- Goals are established (or reinforced if they pre-existed).

- Decisions consistent with these goals are made, consciously or unconsciously, regarding what directions to pursue and what resources to use:

  - If the situation is familiar, then the process may be relatively automatic, where the action(s) taken are in essence the access and implementation of scripts, frames, routines, or schemata.

  - If the situation is not familiar or there is something non-routine about it, then decision-making is made by a mechanism that can be modeled by (i.e., is consistent with the results of) using the subjective expected values of available options, given the orientations of the individual.

- Implementation begins.

- Monitoring (whether effective or not) takes place on an ongoing basis.

- This process is iterative, down to the level of individual utterances or actions:

  - Routines aimed at particular goals have sub-routines, which have their own subgoals;

  - If a subgoal is satisfied, the individual proceeds to another goal or subgoal;

  - If a goal is achieved, new goals kick in via decision-making;

  - If the process is interrupted or things don't seem to be going well, decision-making kicks into action once again. This may or may not result in a change of goals and/or the pathways used to try to achieve them.

Figure 1. How things work, in outline
From Schoenfeld (2010), p. 18, with permission

First Teaching Example, Mark Nelson

Mark Nelson is a beginning teacher. In an elementary algebra class, Nelson has worked through problems like, $x^5/x^3 = $? Now he has assigned

$$\text{(a) } m^6/m^2, \quad \text{(b) } x^3y^7/x^2y^6, \quad \text{and} \quad \text{(c) } x^5/x^5$$

for the class to work. Nelson expects the students to have little trouble with m^6/m^2 and x^3y^7/x^2y^6, but to be "confused" about x^5/x^5; he plans to "work through" their confusion. Here is what happens.

Nelson calls on students to give answers to the first two examples. He has a straightforward method for doing so:

-        He asks the students what they got for the answer, and confirms that it is correct;

-        he asks how they got the answer;

-        and then he elaborates on their responses.

Thus, for example, when a student says the answer to problem (b) is xy, Nelson asks "why did you get xy?" When the student says that he subtracted, Nelson asks, "What did you subtract? When the student says "3 minus 2," Nelson elaborates:

"OK. You looked at the x's [pointing to x-terms in numerator and denominator] and [pointing to exponents] you subtracted 3 minus 2. That gave you x to the first [writes x on the board]. And then [points to y terms] you looked at the y's and said [points to the exponents] 7 minus 6, gives you y to the first [writes y on board]."

He then asks what to do with x^5/x^5. They expand and "cancel." The board shows

$$\frac{x^5}{x^5} = \frac{\cancel{x}\cancel{x}\cancel{x}\cancel{x}\cancel{x}}{\cancel{x}\cancel{x}\cancel{x}\cancel{x}\cancel{x}}.$$ Pointing to that expression, he says, "what do I have?" The responses are

"zero," "zip,", "nada," "nothing" … not what he wants them to see! He tries various ways to get the students to see that "canceling" results in a "1", for example,

Nelson: "What's 5/5?"

Students: "1."

Nelson: "But I cancelled. If there's a 1 there [in 5/5], isn't there a 1 there [pointing to the cancelled expression]?"

Students: "No."

Defeated, he slumps at the board while students argue there's "nothing there." He looks as if there is nothing he can say or do that will make sense to the students.

He tries again. He points to the expression $\frac{x^5}{x^5} = \frac{xxxxx}{xxxxx}$ and asks what the answer is. A student says "*x* to the zero over 1." Interestingly, Nelson *mis-hears* this as "*x* to the zero equals 1," which is the correct answer. Relieved, he tells the class,

"That's right. Get this in your notes: $x^5/x^5 = x^0 = 1$."

Any number to the zero power equals 1."

To put things simply, this is *very* strange. Nelson certainly knew enough mathematics to be able to explain that if x ≠ 0,

$$\left(\frac{x^5}{x^5}\right) = \left(\frac{x}{x}\right)^5 = 1^5 = 1,$$

but he didn't do so. WHY?

There is a simple answer, although it took us a long time to understand it. The issue has to do with Nelson's beliefs and orientations about teaching. One of Nelson's central beliefs about teaching – the belief that *the ideas you discuss must be generated by the students* – shaped what knowledge he did and did not use.

In the first example above (reducing the fraction $x^3y^7/x^2y^{6)}$, a student said he had subtracted. The fact that a student mentioned subtraction gave Nelson "permission" to explain, which he did: "OK. You looked at the x's and you subtracted 3 minus 2. That gave you x to the first. And then you looked at the y's and said 7 minus 6, gives you y to the first."

But in the case of example (c), x^5/x^5, he was stymied – when he pointed to the expression $\frac{xxxxx}{xxxxx}$ and asked "what do I have?" the only answers from the students were "zero," "zip,", "nada," and "nothing." Nobody said "1." And because of his belief that he had to "build on" what students say, Nelson felt he could not proceed with the explanation. Only later, when he *mis-heard* what a student said, was he able to finish up his explanation.

[Note: This brief explanation may or may not seem convincing. I note that full detail is given in the book, and that Nelson was pat of the team that analyzed his videotape. So there is strong evidence that the claims I make here are justified.]

Second Teaching Example, Jim Minstrell

Here too I provide just a very brief description.

Jim Minstrell is an award-winning teacher who is very thoughtful about his teaching. It is the beginning of the school year, and he is teaching an introductory lesson that involves the use of mean, median, and mode. But, the main point of the lesson is that Minstrell wants the students to see that such formulas need to be used *sensibly*.

The previous day eight students measured the width of a table. The values:

106.8; 107.0; 107.0; 107.5; 107.0; 107.0; 106.5; 106.0 cm.

Nelson wants the students to discuss the "best number" to represent the width of the table. His plan is for the lesson to have three parts:

1. Which numbers (all or some?) should they use?

2. How should they combine them?

3. With what precision should they report the answer?

Minstrell gave us a tape of the lesson, which we analyzed. The analysis proceeded in stages. We decomposed the lesson into smaller and smaller "episodes," noting for each episode which goals were present, and observing how transitions corresponded to changes in goals. In this way, we decomposed the entire lesson – starting with the lesson as a whole, and ultimately characterizing what happened on a line-by-line basis. See Figures 2 and 3 (next pages) for an example of analytic detail. Figure 2 shows the whole lesson, and the breaks it into major episodes (lesson segments), each of which has its own internal structure. Most of the lesson was very simple to analyze in this way.

Minstrell has a flexible "script" for each part of the lesson:

- He will raise the issue.

- He will ask the class for a suggestion.

- He will clarify and pursue the student suggestion by asking questions, inserting some content if necessary.

- Once this suggestion has been worked through, he will ask for more suggestions.

- When students run out of ideas, he will either inject more ideas or move to the next part of the lesson.

In this way, the lesson unfolds naturally, and it is easy to "capture" it – see Figure 2 for a "top level" summary of how the lesson unfolded. The episodes in the second and third columns, which correspond to an analysis of the lesson as taught, show that Minstrell did cover the big topics as planned.

In fact, a line-by-line analysis (See Schoenfeld, 1998; 2010) shows that when Minstrell was engaged in dealing with expected subject matter, he followed the "script" described above very closely. So, it is easy to model Minstrell's behavior when he is on familiar ground.
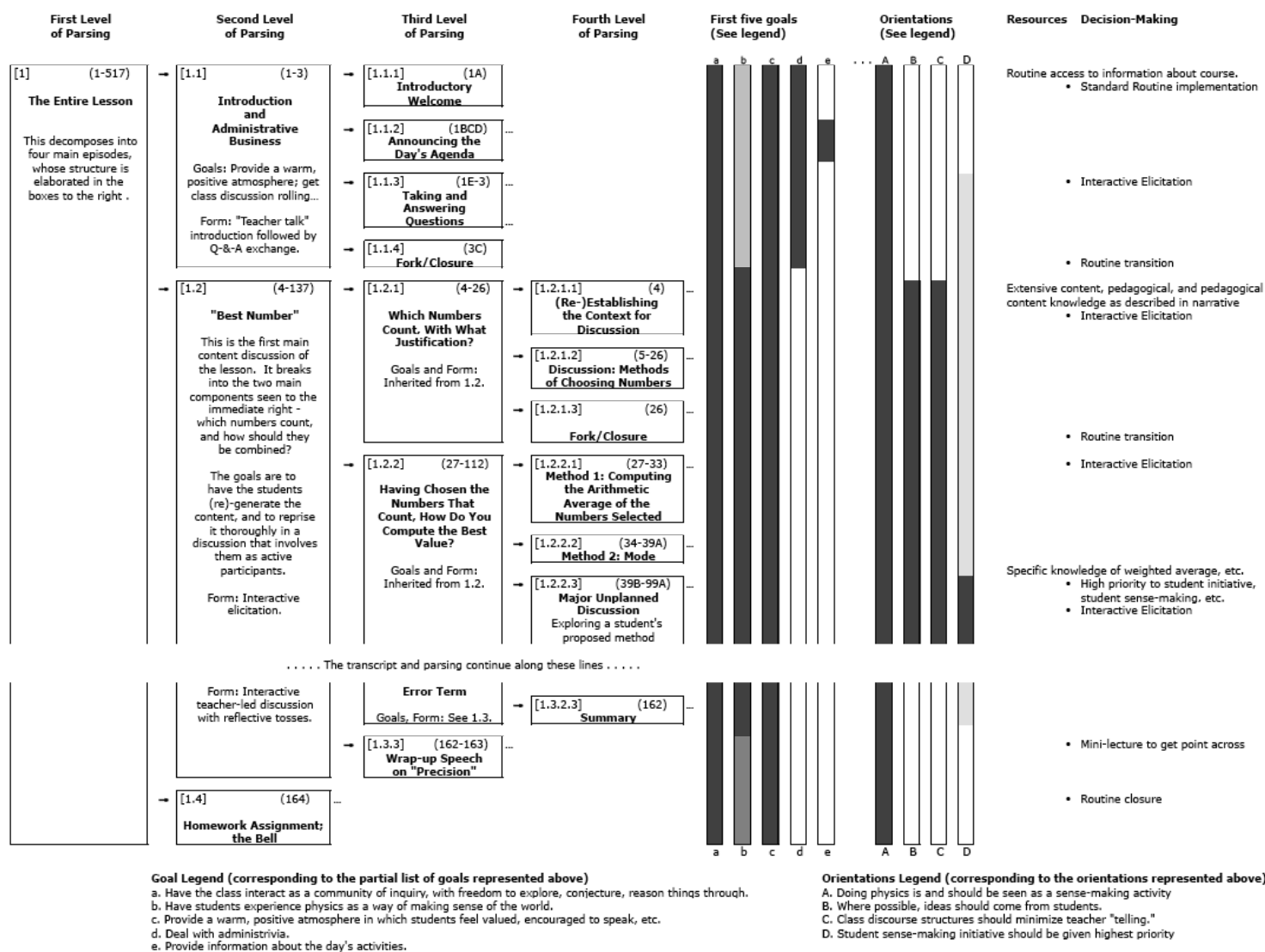
Figure 2. A "top level" view of Minstrell's lesson, "unfolding" in levels of detail.
(With permission, from Schoenfeld, 2010, pp. 96-97)

But what about unusual events? Remember the data: The eight values the students had obtained for the width of the table were

106.8; 107.0; 107.0; 107.5; 107.0; 107.0; 106.5; 106.0 cm.

As the lesson unfolded, Minstrell asked the students about "a way of getting the best value." (See box 1.2.2 in the third column of Figure 2.) As the class proceeded, one student mentioned the idea of using the "average" and, when asked by Minstrell, provided a definition. (Box 1.2.2.1 in the fourth column of Figure 2.) Another student mentioned mode (Box 1.2.2.2). Then, a student said:

"This is a little complicated but I mean it might work. If you see that 107 shows up 4 times, you give it a coefficient of 4, and then 107.5 only shows up one time, you give it a coefficient of one, you add all those up and then you divide by the number of coefficients you have."

This is an unexpected comment, which does not fit directly with Minstrell's flexible script. The question is, can we say what Minstrell would do when something unexpected, like this, arises in the middle of his lesson?

Before proceeding, I want to point out that there is a wide range of responses, which teachers might produce. I have seen responses like all of the following:

-        "That's a very interesting question. I'll talk to you about it after class."

-        "Excellent question. I need to get through today's plans so you can do tonight's assigned homework, but I'll discuss it tomorrow."

-        "That's neat. What you've just described is known as the 'weighted average.' Let me briefly explain how you can work with that…"

-        "Let me write that up as a formula and see what folks think of it."

-        "Let's make sure we all understand what you've suggested, and then explore it."

So, teachers might do very different things. Is it possible to know what Minstrell will do? According to our model of Minstrell,

•   His fundamental *orientation* toward teaching is that physics is a sense-making activity and that students should experience it as such.

•   One of his major *goals* is to support inquiry and to honor student attempts at figuring things out.

•   His *resource base* includes favored techniques such as "reflective tosses" – asking questions that get students to explain/elaborate on what they said.

Thus, the model predicts that he will pursue the last option – making sure that the students understand the issue that the student has raised (including the ambiguity about how you add the coefficients; do you divide by 5 or 8?) and pursuing it. He will do so by asking the students questions and working with the ideas they produce.

This is, in fact, what Minstrell did. Figure 3 shows how that segment of the lesson evolved. It is an elaboration of Box 1.2.2.3 in Figure 2.
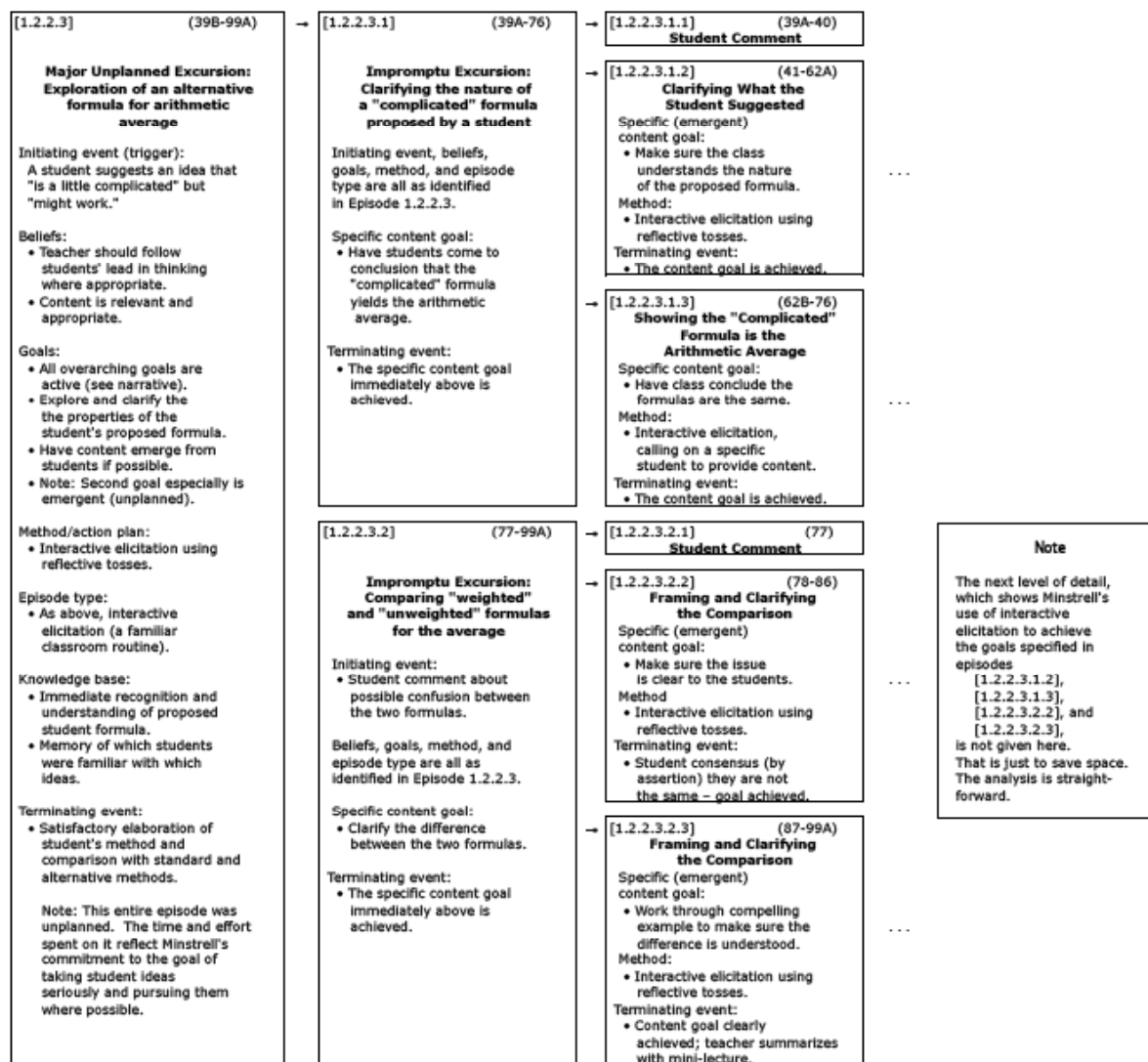
Figure 3. A more fine-grained parsing of Episode [1.2.2.3].
(From Schoenfeld, 2010, pp. 116-117, with permission.)

As noted above, it is possible to model Minstrell's decision, and showing that of the options listed above, he is by far most likely to pursue the one I have indicated. The computations take about 7 pages of text, so I will spare you the detail!

More generally, we have found that:

We were able to capture Minstrell's routine decision-making, on a line-by-line basis, by characterizing his knowledge/resources and modeling them as described in Figure 1, "How Things Work;" and,

We were able to model Minstrell's non-routine decision-making using a form of subjective expected value computation, where we considered the various alternatives and looked at how consistent they were with Minstrell's beliefs and values (his orientations).

In summary, we were able to model every decision Minstrell made during the hour-long class.

Third Teaching Example, Deborah Ball

Some years ago, at a meeting, Deborah Ball showed a video of a third grade classroom lesson she had taught. The lesson was amazing – and it was controversial. In it,

- Third graders argued on solid mathematical grounds.

- The discussion agenda evolved as a function of classroom conversations.

- The teacher seemed at times to play a negligible role, and she made at least one decision that people said didn't make sense.

In addition, I had little or no intuition about what happened. Thus, this was a perfect tape to study! There were major differences from cases 1 and 2:

- the students were third graders instead of high school students

- psychological (developmental) issues differed because of the children's age

- the "control structure" for the classroom was much more "organic"

- the teacher played a less obvious "directing" role.

The question was, could I model what happened in this lesson? If so, then the theory covered an extremely wide range of examples, so there would be compelling evidence of its general validity. If not, then I would understand the limits of the theory. (Perhaps, for example, it would only apply to teacher-directed lessons at the high school level.)

Here is what happened during the lesson. Ball's third grade class had been studying combinations of integers, and they had been thinking about the fact that, for example, the sum of two even numbers always seemed to be even. The previous day Ball's students had met with some 4th graders, to discuss the properties of even numbers, odd numbers, and zero. Ball had wanted her students to see that these were complex issues and that even the "big" fourth graders were struggling with them. The day after the

meeting (the day of this lesson), Ball started the class by asking what the students thought about the meeting:

- How do they think about that experience?

- How do they think about their own thinking and learning?

Ball had students come up to the board to discuss "what they learned from the meeting." The discussion (a transcript of which is given in full in Schoenfeld, 2008, and Schoenfeld, 2010) covered a lot of territory, with Ball seemingly playing a small role as students argued about the properties of zero (is it even? odd? "special"?). For the most part, Ball kept her students on the "meta-level" question: what did they learn about their own thinking from the meeting with the fourth graders the previous day?

But then, after a student made a comment, Ball interrupted him to ask a *mathematical* question about the student's understanding. This question, which took almost 3 minutes to resolve, completely disrupted the flow of the lesson. Many people, when watching the tape of the lesson, call that decision a "mistake." How could Ball, who is a very careful, thoughtful, and experienced teacher, do such a thing? If the decision was arbitrary or capricious in some way, that is a problem for the theory. If highly experienced teachers make arbitrary decisions, it would be impossible to model teachers' decision making in general.

To sum up, this part of the lesson seems to unfold without Ball playing a directive role in its development – and she made an unusual decision to interrupt the flow of conversation. Can this be modeled?

The answer is yes. A fine-grained analysis reveals that Ball has a "debriefing routine" that consists of asking questions and fleshing out answers. That routine is given in Figure 4.

Legend

Action

Decision point

A1 Provide context and background about the topic

A2 Ask class, 'What (else) can you say about the topic?' Call on a student

D1 Does the response raise other issues?

D2 Is clarification called for?

A3 Have the class work through the issues

A4 Seek closure

D3 Is clarification called for?

A5 Either provide or ask student for clarification/ elaboration

D4 Would expansion or reframing be useful?

A6 Highlight particular aspects of discussion for class

D5 Do circumstances warrant more discussion of topic?
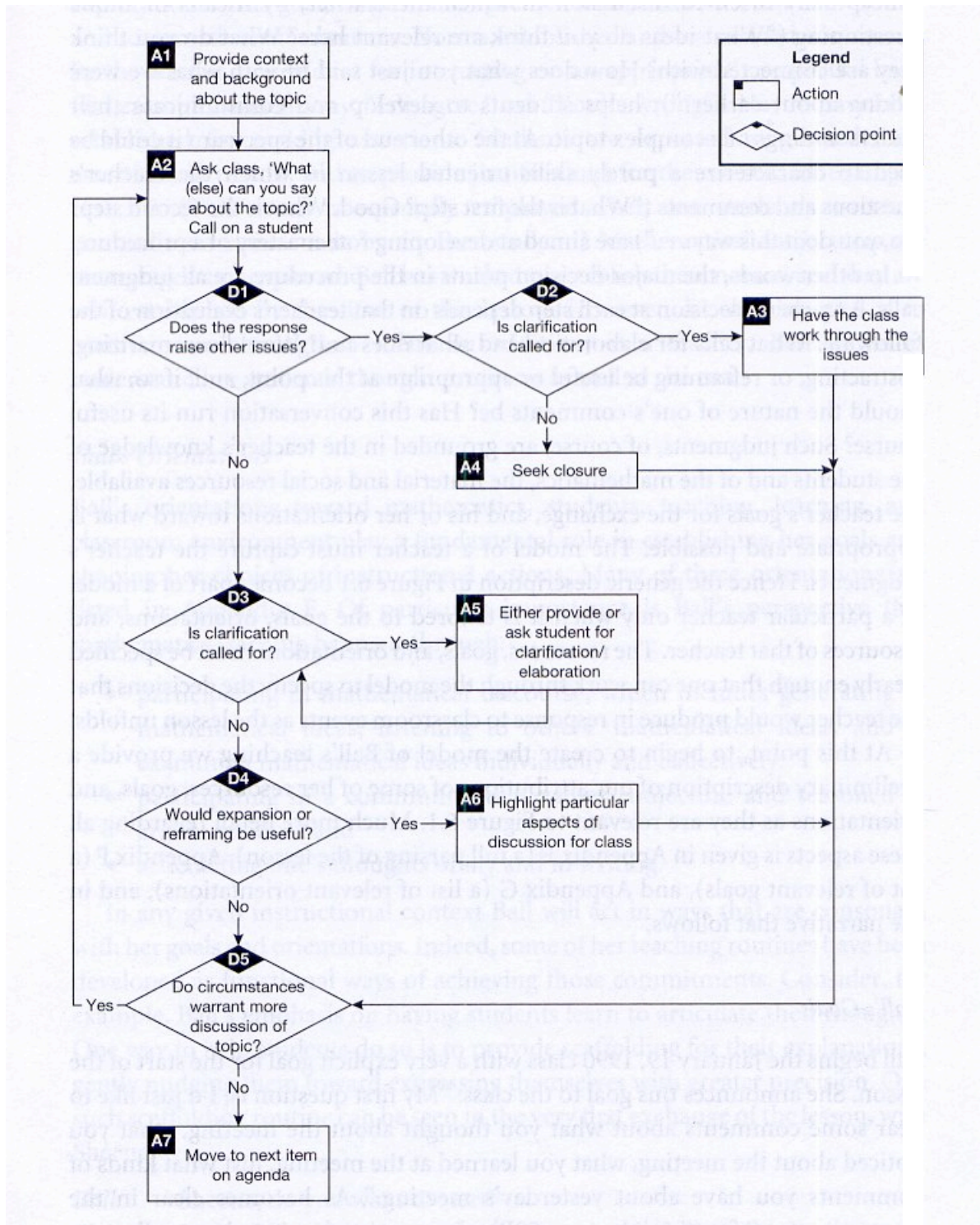
A7 Move to next item on agenda

Figure 4. A flexible, interruptible routine for discussing a topic
(From Schoenfeld, 2010, p. 129, with permission).

In fact, Ball uses that routine five times in the first six minutes of class (Figure 5).
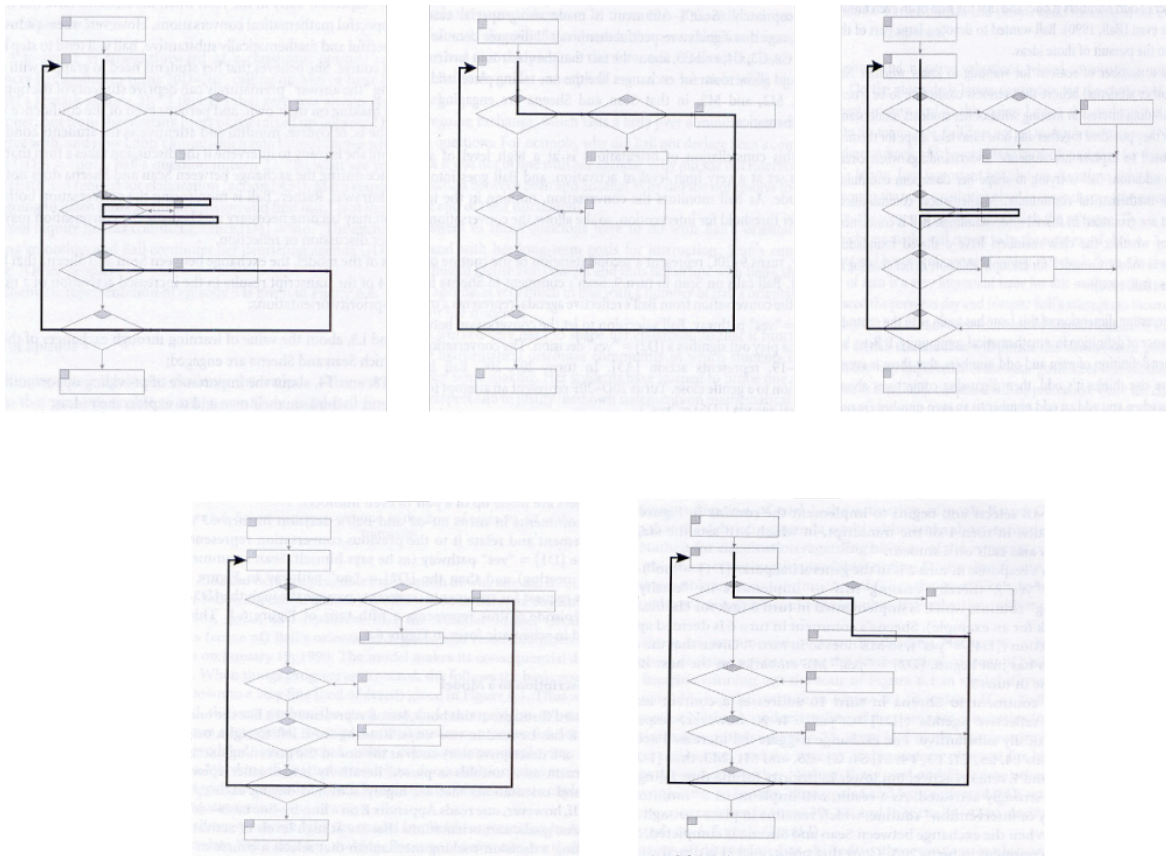
Figure 5. Ball's use of the routine in the lesson segment.
(From Schoenfeld, 2010, pp. 136-152, with permission)

Moreover, once you understand Ball's plans for the lesson, her unexpected decision (her "mistake" can be seen as entirely reasonable and consistent with her agenda. This has been modeled in great detail. For the full analysis, see Schoenfeld, 2010; for an analytic diagram showing the full analysis, download Appendix E from my web page, http://www-gse.berkeley.edu/faculty/AHSchoenfeld/AHSchoenfeld.html.

To sum things up: As in the two previous cases,

We were able to model Ball's routine decision-making, on a line-by-line basis, by characterizing her knowledge/resources and modeling them as described in the slide "How Things Work."

We were able to model Ball's non-routine decision-making as a form of subjective expected value computation.

In short, we were able to model every move Ball made during the lesson segment.

Yet More Examples

Making Breakfast (or any other meal)

If you look at Figure 1, you can see that it would be easy to model decision-making during cooking. Usually we have fixed routines for cooking familiar meals. And if something changes (for example, when my daughter asks me to make a fancy breakfast), that calls for a "non-routine" decision, which can also be modeled.

Routine medical diagnosis and practice

To see if my ideas worked outside of the classroom, I asked my doctor if I could tape and analyze one of my office visits with her. She said yes, and an analysis of our conversation is given in the book. The conversation was easy to model, because she follows a straightforward (and flexible) script. Modeling a two-person interaction is a lot easier than modeling a classroom; it is more like modeling a tutoring interaction. When the teacher only has to pay attention to one other person (instead of 30), decision-making is comparatively simple (and simple to model).

I should note that there is a very large artificial intelligence literature on modeling doctors' decision making – there are computer programs that make diagnoses, etc. (The field is well established: see, e.g., Clancey & Shortliffe, 1984.) So, the idea that it is possible to capture doctors' routine decision making is not new. More recent, and also consistent with my emphasis on beliefs as shaping behavior, there are studies (e.g., Groopman, 2007) of how doctors' stereotypes (orientations) regarding patient behavior lead them to miss what should be straightforward diagnoses.

In Sum:

The approach I have outlined in this paper "covers":

- Routine and non-routine problem solving

- Routine and non-routine teaching

- Cooking

- Brain surgery

and every other example of "well-practiced," knowledge-based behavior that I can think of. All told, I believe it works pretty well as a theory of "how we think."

Readers have the right to ask, why would someone spend 25 years trying to build and test a theory like this? Here is my response.

First, theory building and testing should be a central part of the research enterprise. That is how we make progress.

Second, the more we understand something the better we can make it work. When we understand how something skillful is done we can help others do it.

Third, this approach has the potential to provide tools for describing developmental trajectories of teachers. In that way, we can think about how best to help teachers get better at helping their students learn. (See Chapter 8 of Schoenfeld, 2010.)

Finally, it's fun! The challenge of understanding human behavior has occupied me for the past 35 years, and I am happy to continue working on it.

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